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Investments for the Short and Long Run

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Abstract. This paper aims to discuss the optimal selection of investments for the short and long run in a continuous time financial market setting. First it documents the almost sure pathwise long run outperformance of all positive portfolios by the growth optimal portfolio. Secondly it assumes that every investor prefers more rather than less wealth and keeps the freedom to adjust his or her risk aversion at any time. In a general continuous market, a two fund separation result is derived which yields optimal portfolios located on the Markowitz efficient frontier. An optimal portfolio is shown to have a fraction of its wealth invested in the growth optimal portfolio and the remaining fraction in the savings account. The risk aversion of the investor at a given time determines the volatility of her or his optimal portfolio. It is pointed out that it is usually not rational to reduce risk aversion further than is necessary to achieve the maximum growth rate. Assuming an optimal dynamics for a global market, the market portfolio turns out to be growth optimal. The discounted market portfolio is shown to follow a particular time transformed diffusion process with explicitly known transition density. Assuming that the transformed time growth exponentially, a parsimonious and realistic model for the market portfolio dynamics results. It allows for efficient portfolio optimization and derivative pricing.

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1 Introduction

Throughout recent decades there has been an ongoing debate on the following important question: If an investor invests for the “long run”, this means with still several decades to go, is the growth optimal portfolio an appropriate choice? This portfolio maximizes the expected logarithm of the wealth accumulated during the long investment period ahead. There seem to be at least two answers suggested in the literature. One is given in Paul Samuelson’s well-known note, see Samuelson (1979), which in a remarkably, strong manner states “why we should not make mean log of wealth though years to act are long”, see also Samuelson (1963, 1969, 1979). Other authors, including Markowitz (1959, 1976), Latané (1959), Breiman (1961), Thorp (1972), Hakansson (1971), Rubinstein (1976), Cover (1991), Ziemba & Mulvey (1998), Browne (1999) and Stutzer (2000), see the maximization of the growth rate of a portfolio by an investor with an extremely long time horizon as an acceptable strategy.

This paper aims to outline theoretical reasons for the conclusion that for an investor with an extremely long time horizon, the answer should be in favor of choosing the growth optimal strategy. The growth optimal strategy should therefore guide pension funds, insurance companies and other long term investors. However, for an investor with a rather limited time horizon the answer is more subtle. An investor who prefers more rather than less wealth and keeps the freedom to adjust his or her risk aversion at any time will be shown to select optimally a mix between the growth optimal portfolio and the savings account. Similarly, we will show that for the optimal market dynamics, an investor who maximizes expected utility from terminal wealth will also invest a certain fraction of wealth in the *growth optimal portfolio* (GOP) at any time, with the remainder in the savings account. This two fund separation result, see Tobin (1958) and Sharpe (1964), can be obtained very generally and yields efficient portfolios in the sense of Markowitz (1959). We will argue that in an optimal portfolio it usually makes no sense to lower the risk aversion further than is required to obtain the maximum growth rate, otherwise unnecessary risk is taken.

In Markowitz (1959, 1976) it has been explained why the GOP has an important role to play as an investment vehicle. Of course, the fluctuations of this portfolio can be substantial. By assuming that all investors form optimal portfolios it will be shown to equal the market portfolio. It is a key mathematical property of the GOP that it almost surely outperforms any other strictly positive portfolio after sufficient long time. Furthermore, it will be demonstrated that the discounted GOP has a very suggestive dynamics, after a natural time transformation. By assuming that the transformed time growth exponentially, the minimal market model, see Platen (2001, 2002), emerges.

The outline of the paper is as follows: Section 2 describes important properties of the GOP for a general financial market. Section 3 derives the GOP for a general continuous market. Section 4 discusses optimal portfolio selection. Properties of

optimal portfolios are discussed in Section 5. The dynamics of the discounted market portfolio are studied in Section 6. In Section 7 utility maximization is discussed.

2 Continuous Time Financial Market

We consider a financial market in continuous time with $d + 1$ *primary security accounts*, $d \in \{1, 2, \dots\}$. These are typically stocks or savings accounts of different currencies, where all income is reinvested. In the case of stocks, S_t^j denotes the cum-dividend value of the j th stock at time $t \in [0, \infty)$, $j \in \{0, 1, \dots, d\}$. Assume that the units of primary security accounts are infinitely divisible and that continuous frictionless trading is possible.

The dynamics of the primary security account vector process $S = \{S_t = (S_t^0, S_t^1, \dots, S_t^d)^\top, t \in [0, \infty)\}$ is allowed to be very general, covering most discrete time and continuous time models. For the mathematically interested reader, we mention that S can be a semimartingale on a filtered probability space $(\Omega, \mathcal{A}, \underline{\mathcal{A}}, P)$, satisfying the usual conditions, see Protter (2004). Here $\underline{\mathcal{A}} = (\mathcal{A}_t)_{t \in [0, \infty)}$, where \mathcal{A}_t represents the *information* available at time t .

In addition to the given $d + 1$ primary security accounts, we consider *portfolios*. A portfolio value S_t^δ at time t is a linear combination of primary security accounts:

$$S_t^\delta = \delta_t^\top S_t = \sum_{j=0}^d \delta_t^j S_t^j \quad (2.1)$$

for all $t \in [0, \infty)$. Here the *strategy* $\delta = \{\delta_t = (\delta_t^0, \delta_t^1, \dots, \delta_t^d)^\top, t \in [0, \infty)\}$ describes with its j th component δ_t^j the number of units of the j th primary security account that are held at time t in the portfolio S^δ . The portfolios we consider are assumed to be *self-financing*, which means that changes in portfolio values are due only to gains from trade.

An important investment indicator is the *expected rate of return* of a portfolio. Clearly, this quantity depends on the underlying denomination. For instance, one could use the domestic savings account, the market portfolio, or any other strictly positive portfolio process as the unit in which to denominate a given portfolio. One expects a realistic financial market to contain a strictly positive portfolio, which when used as reference unit or benchmark forces the expected rates of return of all benchmarked portfolios to be finite. Therefore, let us introduce the following extremely weak assumption, where $E(\cdot | \mathcal{A}_t)$ denotes the conditional expectation under the historical probability measure P , given the information \mathcal{A}_t at time $t \in [0, \infty)$.

Assumption 2.1 *Assume that for each time $\tau \in [0, \infty)$ there exists a strictly positive portfolio $S^{\delta+}$ and a nonnegative random variable $K_\tau^{\delta+} \in [0, \infty)$ with*

$E(K_\tau^{\delta+} \mid \mathcal{A}_0) < \infty$ such that for all times $\sigma \in (\tau, \infty)$ and all nonnegative portfolios S^δ we have

$$\frac{1}{\sigma - \tau} E \left(\frac{\frac{S_\sigma^\delta}{S_{\sigma+}^{\delta+}} - \frac{S_\tau^\delta}{S_{\tau+}^{\delta+}}}{\frac{S_\tau^\delta}{S_{\tau+}^{\delta+}}} \mid \mathcal{A}_\tau \right) \leq K_\tau^{\delta+}. \quad (2.2)$$

Let us introduce another important investment indicator. For a pair of times $\tau \in [0, \infty)$ and $\sigma \in (\tau, \infty)$ and a given strictly positive portfolio process S^δ its *expected growth rate* $g_{\tau, \sigma}^\delta$ over the period $[\tau, \sigma]$ is defined as the conditional expectation

$$g_{\tau, \sigma}^\delta = \frac{1}{\sigma - \tau} E \left(\ln \left(\frac{S_\sigma^\delta}{S_\tau^\delta} \right) \mid \mathcal{A}_\tau \right). \quad (2.3)$$

This notion allows us to introduce a *growth optimal portfolio* (GOP) $S^{(\delta^*)}$, which is defined as a portfolio with maximum expected growth rate over all finite time periods $[\tau, \sigma]$, that is, one has

$$g_{\tau, \sigma}^\delta \leq g_{\tau, \sigma}^{\delta^*} \quad (2.4)$$

almost surely for all strictly positive portfolios S^δ and all times $0 \leq \tau < \sigma < \infty$. It is worth mentioning that due to (2.3) and (2.4) the composition of a GOP does *not* depend on the denomination of the portfolio.

For the given general class of financial market models it has been shown in Platen (2004a) that a GOP exists under Assumption 2.1. It equals $S^{\delta+}$ if and only if in (2.2) the random variable $K_\tau^{\delta+}$ can be set to zero for all $\tau \in (0, \infty)$. In this case the reference portfolio $S^{\delta+}$ is a GOP. For any nonnegative portfolio S^δ the existence of a GOP S^{δ^*} leads by (2.2) to the relationship

$$E \left(\frac{S_\sigma^\delta}{S_{\sigma+}^{\delta^*}} \mid \mathcal{A}_\tau \right) \leq \frac{S_\tau^\delta}{S_{\tau+}^{\delta^*}} \quad (2.5)$$

for all $0 \leq \tau < \sigma < \infty$. Inequality (2.5) demonstrates that a GOP S^{δ^*} outperforms all nonnegative portfolios in the sense that when S^δ is expressed in units of S^{δ^*} it yields at most zero expected rates of return. This indicates that the GOP is the best performing portfolio, not only in terms of expected growth rate, see (2.4), but also in the sense of maximizing the expected rate of return. It should be mentioned that relation (2.5) excludes a weak but very natural form of arbitrage related to the limited liability of each investor, as demonstrated in Platen (2004a).

Let us introduce for any strictly positive portfolio S^δ its *long term growth rate* \tilde{g}_∞^δ . To be mathematically precise we define it as the almost sure upper limit

$$\tilde{g}_\infty^\delta \stackrel{\text{a.s.}}{=} \limsup_{t \rightarrow \infty} \frac{1}{t} \ln \left(\frac{S_t^\delta}{S_0^\delta} \right). \quad (2.6)$$

Intuitively, this quantity describes the average slope of the logarithm of a portfolio when the time horizon becomes extremely large. For the given general market

model it has been shown in Platen (2004a) that the GOP $S^{\delta*}$ provides the maximum long term growth rate, such that

$$\tilde{g}_{\infty}^{\delta*} \geq \tilde{g}_{\infty}^{\delta} \quad (2.7)$$

almost surely, for all strictly positive portfolios S^{δ} . This generalizes results in Kelly (1956), Thorp (1972) and Karatzas & Shreve (1998). The property (2.7) states that the GOP almost surely *outperforms* all other strictly positive portfolios *in the long run*. It must be emphasized that no assumptions are imposed that limit the practical relevance of this result. Therefore, this is a model independent property of the GOP that makes this portfolio unique and extremely attractive for long term investors.

If an investor has practically no time limitation, then a GOP is clearly the portfolio of choice. This is likely to apply to pension funds, insurance funds and potentially for young individuals. Samuelson warned in his 1979 note that this portfolio may be too risky for individuals. This is certainly true when the retirement horizon is close. The question of what portfolio choice is appropriate if there are only a few years to go will be discussed below.

3 Continuous Financial Market

To investigate in more detail which type of portfolio short and medium term investors should select it is convenient to consider a general continuous financial market where primary security account prices are continuous over time. We remark that similar results can also be derived for markets with jumps, see Platen (2004b). A particular simple version of such a market model was pioneered in Merton (1973b) and Black & Scholes (1973) with the widely used Black-Scholes model. Note however that the following results apply for any market model with continuous securities.

Consider d continuous sources of trading uncertainty, which are modeled by the d -dimensional standard Wiener process $W = \{W_t = (W_t^1, W_t^2, \dots, W_t^d)^{\top}, t \in [0, \infty)\}$, $d \in \{1, 2, \dots\}$. The j th primary security account is assumed to be the solution of the Itô stochastic differential equation (SDE)

$$dS_t^j = S_t^j \left(a_t^j dt + \sum_{k=1}^d b_t^{j,k} dW_t^k \right) \quad (3.1)$$

for all $t \in [0, \infty)$, with $S_0^j > 0$ and $j \in \{1, 2, \dots, d\}$, see Karatzas & Shreve (1991).

We introduce a locally riskless *savings account* $S^0 = \{S_t^0, t \in [0, \infty)\}$, with

$$S_t^0 = \exp \left\{ \int_0^t r_s ds \right\}. \quad (3.2)$$

The *appreciation rate* processes $a^j = \{a_t^j, t \in [0, \infty)\}$, short rate process $r = \{r_t, t \in [0, \infty)\}$ and *volatility* processes $b^{j,k} = \{b_t^{j,k}, t \in [0, \infty)\}$ are allowed to be general stochastic processes for all $j, k \in \{1, 2, \dots, d\}$. It is only required that a unique solution of the SDE (3.1) exists. Note that the appreciation rates are almost sure limits of expected rates of return over time periods with decreasing length.

To avoid arbitrage, see Platen (2002), the following natural condition is imposed:

Assumption 3.1 *In the given continuous market model the volatility matrix $b_t = [b_t^{j,k}]_{j,k=1}^d$ is invertible for each $t \in [0, \infty)$.*

By using the appreciation rate vector $a_t = (a_t^1, a_t^2, \dots, a_t^d)^\top$ and the unit vector $\mathbf{1} = (1, 1, \dots, 1)^\top$ one can introduce the *market price of risk* vector

$$\theta_t = (\theta_t^1, \theta_t^2, \dots, \theta_t^d)^\top = b_t^{-1} [a_t - r_t \mathbf{1}] \quad (3.3)$$

at the time $t \in [0, \infty)$. This allows one to rewrite the SDE (3.1) for the j th primary security account S_t^j in the form

$$dS_t^j = S_t^j \left(r_t dt + \sum_{k=1}^d b_t^{j,k} (\theta_t^k dt + dW_t^k) \right) \quad (3.4)$$

for $t \in [0, \infty)$, $j \in \{1, 2, \dots, d\}$.

For a given strategy $\delta = \{\delta_t = (\delta_t^0, \delta_t^1, \dots, \delta_t^d)^\top, t \in [0, T]\}$ with δ_t^j units of the j th primary security invested at time t in the j th primary security account, it is convenient to introduce the j th *fraction* of the value of the corresponding strictly positive portfolio S_t^δ as follows:

$$\pi_{\delta,t}^j = \delta_t^j \frac{S_t^j}{S_t^\delta} \quad (3.5)$$

for $t \in [0, \infty)$ and $j \in \{0, 1, \dots, d\}$. Note that the fractions always sum to unity. For a given strictly positive, self financing portfolio value S_t^δ we then obtain from (2.1), (3.4) and (3.5) the SDE

$$dS_t^\delta = S_t^\delta \left(r_t dt + \sum_{k=1}^d \sum_{j=1}^d \pi_{\delta,t}^j b_t^{j,k} (\theta_t^k dt + dW_t^k) \right) \quad (3.6)$$

for $t \in [0, \infty)$. From (3.6) it follows by application of the Itô formula, see Karatzas & Shreve (1998), that the logarithm of a strictly positive portfolio S_t^δ satisfies the SDE

$$d \ln(S_t^\delta) = g_t^\delta dt + \sum_{k=1}^d \sum_{j=1}^d \pi_{\delta,t}^j b_t^{j,k} dW_t^k \quad (3.7)$$

with *growth rate*

$$g_t^\delta \stackrel{\text{a.s.}}{=} \lim_{h \rightarrow 0} g_{t,t+h}^\delta \stackrel{\text{a.s.}}{=} r_t + \sum_{k=1}^d \sum_{j=1}^d \pi_{\delta,t}^j b_t^{j,k} \left(\theta_t^k - \frac{1}{2} \sum_{\ell=1}^d \pi_{\delta,t}^\ell b_t^{\ell,k} \right) \quad (3.8)$$

for all $t \in [0, \infty)$. Note that the growth rate in the given continuous market is the almost sure limit of the expected growth rate (2.3) over time periods with decreasing length.

The GOP $S_t^{\delta^*}$ is then obtained via (2.4) and (3.8) by maximizing the above growth rate, so that

$$g_t^{\delta^*} \geq g_t^\delta \quad (3.9)$$

for all strictly positive portfolio processes S^δ and $t \in [0, \infty)$. This means, to identify the GOP we have to maximize a quadratic form (3.8) with respect to the fractions introduced in (3.5). For each $t \in [0, \infty)$ this quadratic optimization problem has a unique solution that is characterized by the first order condition

$$0 = \sum_{k=1}^d b_t^{j,k} \left(\theta_t^k - \sum_{\ell=1}^d \pi_{\delta,t}^\ell b_t^{\ell,k} \right) \quad (3.10)$$

for all $j \in \{1, 2, \dots, d\}$, see, for instance, Merton (1973a). The vector of optimal fractions for the GOP is then of the form

$$\pi_{\delta^*,t} = (\pi_{\delta^*,t}^1, \dots, \pi_{\delta^*,t}^d)^\top = (b_t^{-1})^\top \theta_t \quad (3.11)$$

for all $t \in [0, \infty)$. Consequently, by (3.11) and (3.6) the value $S_t^{\delta^*}$ of the GOP satisfies the SDE

$$dS_t^{\delta^*} = S_t^{\delta^*} \left(\left[r_t + \sum_{k=1}^d (\theta_t^k)^2 \right] dt + \sum_{k=1}^d \theta_t^k dW_t^k \right) \quad (3.12)$$

for $t \in [0, \infty)$, where $S_0^{\delta^*} > 0$. One notes that only the market prices of risk enter the SDE of the discounted GOP when it is discounted by the savings account. Furthermore, the drift is determined by the diffusion coefficient. This link between drift and diffusion coefficient is an important consequence of the optimization of the growth rate and reduces enormously the complexity of the dynamics under consideration.

4 Portfolio Selection

In the above continuous market model the short rate, volatilities and market prices of risk are flexible stochastic processes. They provide substantial freedom for realistic modeling.

Let us now investigate the strategy likely to be selected by an investor who prefers more rather than less wealth and keeps the freedom to adjust her or his investment strategy at any time, according to changing personal circumstances or incoming information. We assume that the investor takes the time value of money into account by considering *discounted portfolio* values

$$\bar{S}_t^\delta = \frac{S_t^\delta}{S_t^0}, \quad (4.1)$$

$t \in [0, \infty)$. By (4.1), (3.6) and application of the Itô formula we obtain for the discounted portfolio value \bar{S}_t^δ the following SDE

$$d\bar{S}_t^\delta = \psi_{\delta,t}^\top \{\theta_t dt + dW_t\}, \quad (4.2)$$

with *diffusion coefficient* vector

$$\psi_{\delta,t}^\top = (\psi_{\delta,t}^1, \dots, \psi_{\delta,t}^d) = \bar{S}_t^\delta \pi_{\delta,t}^\top b_t \quad (4.3)$$

for $t \in [0, \infty)$. According to (4.2), the *drift* of \bar{S}_t^δ is

$$\alpha_t^\delta = \psi_{\delta,t}^\top \theta_t \quad (4.4)$$

for $t \in [0, \infty)$. This is the trend of \bar{S}_t^δ and models the increase of the *underlying value* of \bar{S}^δ per unit of time. The *aggregate diffusion coefficient* of (4.2) has the form

$$\gamma_t^\delta = \sqrt{\psi_{\delta,t}^\top \psi_{\delta,t}}. \quad (4.5)$$

Its square measures the variance per unit of time of the fluctuating increments of \bar{S}^δ . In some sense, γ_t^δ measures the risk associated with \bar{S}_t^δ locally in time, whereas α_t^δ expresses its trend.

Let us now define a class of *optimal portfolios* that capture the natural objective of investors who prefer to gain more rather than less wealth locally in time. A strictly positive portfolio process \bar{S}^δ is called *optimal* if for all times $t \in [0, \infty)$ and any strictly positive portfolio \bar{S}^δ with the same aggregate diffusion coefficient value

$$\gamma_t^\delta = \gamma_t^{\bar{\delta}} \quad (4.6)$$

it achieves the largest drift, that is

$$\alpha_t^\delta \leq \alpha_t^{\bar{\delta}}. \quad (4.7)$$

This type of optimality can be interpreted as a continuous time generalization of mean-variance optimality in the sense of Markowitz (1952, 1959). A discounted optimal portfolio exhibits the largest trend in comparison with all other discounted positive portfolios with the same risk level at all times. Equivalently, one can say that a discounted optimal portfolio achieves the largest risk premium

if compared with all other discounted positive portfolios with the same aggregate volatility.

The *total market price of risk*

$$|\theta_t| = \sqrt{\theta_t^\top \theta_t} \quad (4.8)$$

is the volatility of the GOP by (3.12). One can show that for zero total market price of risk all portfolios are optimal. To avoid such unrealistic risk neutral market dynamics we introduce the following assumption.

Assumption 4.1 For all $t \in [0, \infty)$ the total market price of risk *satisfies*

$$|\theta_t| > 0 \quad (4.9)$$

while for the fraction of GOP wealth in the savings account we have

$$\pi_{\delta^*,t}^0 \neq 1. \quad (4.10)$$

Let us now identify the SDE for a discounted optimal portfolio. Results on portfolio selection and two fund separation have been obtained by various authors, including Tobin (1958), Breiman (1960), Sharpe (1964), Merton (1973a) and Khanna & Kulldorff (1999). In Platen (2002, 2005a) these have been generalized for continuous market models, where it has been shown that an optimal portfolio S^δ can be parameterized by the fraction $\pi_{\delta,t}^0$ invested in the savings account. The discounted value \bar{S}_t^δ of an optimal portfolio then satisfies the SDE

$$d\bar{S}_t^\delta = \bar{S}_t^\delta (J_t^\delta)^{-1} \theta_t^\top (\theta_t dt + dW_t) \quad (4.11)$$

and the vector of optimal fractions is of the form

$$\pi_{\delta,t} = (\pi_{\delta,t}^1, \dots, \pi_{\delta,t}^d)^\top = (J_t^\delta)^{-1} \pi_{\delta^*,t} \quad (4.12)$$

for $t \in [0, \infty)$, see Platen (2002). Here the ratio

$$J_t^\delta = \frac{1 - \pi_{\delta^*,t}^0}{1 - \pi_{\delta,t}^0} \quad (4.13)$$

can be interpreted as *risk aversion coefficient* in the sense of Arrow (1965) and Pratt (1964). The risk aversion coefficient for the GOP equals one and that for the savins account infinity. We emphasize that the fractions for the investments in risky primary security accounts in (4.12) equal those of the GOP up to a common factor and the result holds very generally for continuous markets.

The fact that the SDE (4.11) for discounted optimal portfolios depends on $\pi_{\delta,t}^0$ is interesting and not self-evident. As shown in Platen (2002), the above result requires no more than multivariate calculus and a basic understanding of Itô

calculus. By (4.12) an optimal portfolio follows a strategy where at each time a fraction of wealth is invested in the GOP and the remainder is held in the savings account. Therefore, the only two funds that an investor needs to consider are the GOP and the savings account, thus yielding two fund separation. The risk aversion of an investor at a given time determines the corresponding optimal portfolio.

The literature usually considers the savings account and a mutual fund, consisting of risky primary security account investments proportional to the fractions of the GOP. This yields an equivalent description of the above two fund separation, but does not acknowledge the central role of the GOP in portfolio optimization, as we will see later.

5 Properties of Optimal Portfolios

For an optimal portfolio S^δ it follows by (4.11) that its *aggregate volatility* is given by

$$|b_\delta(t)| = (J_t^\delta)^{-1} |\theta_t| \quad (5.1)$$

and its risk premium takes the form

$$p_\delta(t) = \frac{\alpha_t^\delta}{\bar{S}_t^\delta} = (J_t^\delta)^{-1} |\theta_t|^2 = |b_\delta(t)| |\theta_t| \quad (5.2)$$

for $t \in [0, \infty)$, see (4.3)–(4.4). Obviously, by (4.11), (5.1) and (5.2) the *appreciation rate* $a_\delta(t)$ of an optimal portfolio can be written as a function of its squared volatility $|b_\delta(t)|^2$ in the form

$$a_\delta(t) = r_t + p_\delta(t) = r_t + |b_\delta(t)| |\theta_t| = r_t + \sqrt{|b_\delta(t)|^2} |\theta_t| \quad (5.3)$$

for $t \in [0, \infty)$. This function can be interpreted as a continuous time version of the *Markowitz efficient frontier*, see Markowitz (1952, 1959). By the form of (5.3) it turns out that each optimal portfolio S^δ is instantaneously efficient in the sense of Markowitz. Its appreciation rate is located on the efficient frontier (5.3). The total market price of risk $|\theta_t|$ and the short rate r_t are the central invariants determining the evolution of the efficient frontier through time. For a fixed time instant $t \in [0, \infty)$ the Figure 1 shows the efficient frontier, that is, the instantaneous appreciation rates of optimal portfolios as functions of the squared volatility $|b_\delta(t)|^2$. The values $r_t = 0.05$ and $|\theta_t|^2 = 0.04$ are used for illustration. In Figure 1 we have also included the tangent with slope $\frac{1}{2}$ at the point $|b_\delta(t)|^2 = |\theta_t|^2$, the squared volatility of the GOP. We will show later in Figure 6 that the efficient frontier moves randomly up and down over time according to the fluctuations of the short rate r_t and the total market price of risk $|\theta_t|$.

If one considers the dependence of the appreciation rate $a_\delta(t)$ in (5.3) on the absolute volatility $|b_\delta(t)|$ of an optimal portfolio, then the total market price of

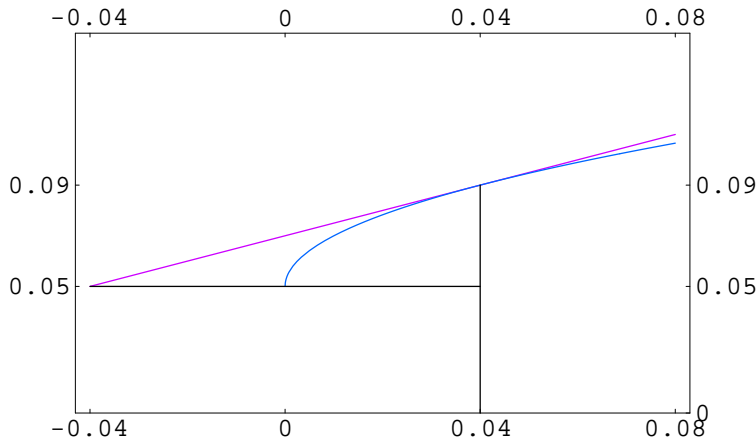


Figure 1: Efficient frontier.

risk $|\theta_t|$ determines the slope of the resulting linear function. In this way the well-known *capital market line* emerges, see Sharpe (1964).

For a risky portfolio S^δ , let

$$s_t^\delta = \frac{p_\delta(t)}{b_\delta(t)} \quad (5.4)$$

denote its *Sharpe ratio*, see Sharpe (1964). It follows from the derivation of the optimal portfolio SDE (4.11) in Platen (2002) that the Sharpe ratio is never greater than that of a risky optimal portfolio, which in turn equals the total market price of risk. One can say that optimal portfolios maximize the Sharpe ratio in a continuous market model.

Let us consider the growth rate

$$g_\delta(t) = r_t + \sqrt{|b_\delta(t)|^2} |\theta_t| - \frac{1}{2} |b_\delta(t)|^2 \quad (5.5)$$

of an optimal portfolio as a function of its squared volatility, see (3.8), (4.12) and (5.1). For illustration, Figure 2 expresses this relationship using the same parameters as in Figure 1, at some time $t \in [0, \infty)$. One notes that for the choice $|b_\delta(t)|^2 = |\theta_t|^2$, the growth rate is maximized. The resulting maximum growth rate

$$g_{\delta^*}(t) = r_t + \frac{1}{2} |\theta_t|^2 \quad (5.6)$$

is that of the GOP. As can be seen from Figure 2, for an optimal portfolio S^δ with $|b_\delta(t)| > |\theta_t|$ the growth rate is less than that of the GOP. For a reasonable investor it is therefore not rational to accept a higher portfolio volatility than that of the GOP. In the long term such a portfolio cannot outperform the GOP, but attracts unnecessary risk. There is no obvious reason for investing in a portfolio with volatility greater than $|\theta_t|$ unless one has several lives or enjoys gambling. Of course, tax advantages may provide a valid reason to borrow funds to invest

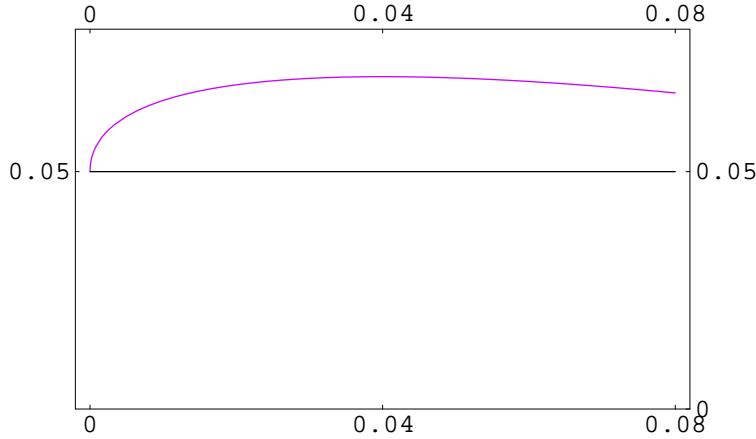


Figure 2: Growth rates of optimal portfolios.

in an optimal portfolio with volatility greater than that of the GOP, however, this is beyond the market model studied in this paper.

One notes that the curve in Figure 2 is rather flat near the maximum growth rate. Therefore, in the region of the optimal growth rate, an investor can reduce her or his optimal portfolio volatility without compromising too much in terms of long term growth. As in the case with the efficient frontier, the curve in Figure 2 evolves over time according to the changing short rate and fluctuating total market price of risk.

The *risk premium* $p_\delta(t)$ of a strictly positive portfolio S^δ has the form

$$p_\delta(t) = \sum_{k=1}^d \sum_{j=1}^d \pi_{\delta,t}^j b_t^{j,k} \theta_t^k \quad (5.7)$$

for $t \in [0, \infty)$, by (3.6). Let $\langle X, Y \rangle_t$ denote the covariation of two continuous stochastic processes. This may be defined as the limiting sum of the products of increments in X and Y based on a time discretization with vanishing step size, see Karatzas & Shreve (1991). It can also be thought of as the continuous time equivalent of the conditional covariance of log-returns. The *systematic risk parameter* $\beta_\delta(t)$ of a portfolio S^δ can then be defined as

$$\beta_\delta(t) = \frac{\frac{d}{dt} \langle \ln(S^\delta), \ln(S^{\delta_{\text{opt}}}) \rangle_t}{\frac{d}{dt} \langle \ln(S^{\delta_{\text{opt}}}), \ln(S^{\delta_{\text{opt}}}) \rangle_t} \quad (5.8)$$

for $t \in [0, T]$, where $S^{\delta_{\text{opt}}}$ is a given risky optimal portfolio. Very generally it then follows directly by (4.11) and (5.7) that, for any strictly positive portfolio S^δ , one has the equation

$$p_\delta(t) = \beta_\delta(t) p_{\delta_{\text{opt}}}(t) \quad (5.9)$$

for $t \in [0, \infty)$, see Platen (2005a). This is the key relationship of the *capital asset pricing model* (CAPM) developed by Sharpe (1964), Lintner (1965), Mossin

(1966) and Merton (1973a). In our setup, this important equation follows directly, so long as one uses a risky optimal portfolio as reference portfolio.

In the CAPM the market portfolio is used as reference unit. To identify the optimal dynamics of the market portfolio let us make the following natural assumption.

Assumption 5.1 *Each investor forms an optimal portfolio with her or his total investable wealth.*

It is straightforward to show, see Platen (2005a), that the sum of optimal portfolios is an optimal portfolio. Therefore, the market portfolio of investable wealth is an optimal portfolio under Assumption 5.1. By equation (5.9) the CAPM therefore holds quite generally. Note that the derivation of this result does not require any assumptions about equilibrium, expected utility or Markovianity, as typically imposed in the literature. Obviously, risk premia and betas change randomly over time, as does the efficient frontier. This makes it difficult to estimate these quantities. One needs a proper understanding of the probabilistic nature of the dynamics of the market portfolio to estimate the parameters that control its evolution, a problem that we address in the next section. The main reason why empirical verifications of the CAPM and the estimation of betas usually fail in the literature appears to be that the analysis is typically based on a Black-Scholes type market model. The geometric Brownian motion dynamics of the Black-Scholes model are mathematically convenient and very tractable. However, there is no economic reason why the optimal market dynamics should follow the lognormal paradigm.

6 Dynamics of the Discounted Market Portfolio

To identify the optimal composition of the market portfolio we acknowledge the global nature of the world market by introducing the following assumption:

Assumption 6.1 *The fundamental relationships in the market are invariant under changes of currency denomination.*

By this condition it follows from our previous results that the market portfolio of investable wealth must be optimal under all currency denominations. In Platen (2005b) it has been shown that there is only one portfolio with this property, namely the GOP. This provides us with the important insight that under optimal market dynamics the *market portfolio has to be a GOP*. For illustration, Figure 3 shows the logarithm of the discounted MSCI *world stock index* (MSCI), expressed in units of the US dollar savings account, from 1970 until 2003. It can be regarded

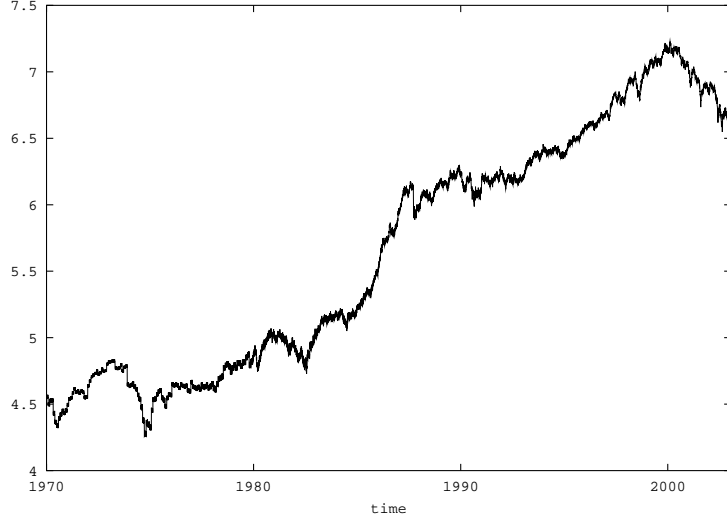


Figure 3: Logarithm of discounted MSCI.

as a reasonable proxy for the logarithm of the discounted GOP of the world stock market. From our previous discussion on two fund separation, it is optimal for an investor to invest in the market portfolio and some nonnegative fraction of his or her wealth in the savings account. If his or her risk aversion is high, then a large fraction of wealth needs to be invested in the savings account. If the investor has little concern about risk, then the market portfolio is the appropriate investment, which is pathwise the best investment for the long run since it represents the GOP.

Instead of using volatility as parameter process, as is the case under the lognormal paradigm based on the Black-Scholes model, we take now the drift of the discounted GOP as parameter process. This makes economic sense because this parameter process reflects the increase per unit of time of underlying discounted value of the market portfolio. Since underlying economic value appears to accumulate slowly but steadily over time in the world economy, one can expect this quantity to evolve rather smoothly. This makes it potentially a better suited parameter process for asset modeling than volatility. By (3.12) and (4.1) the discounted GOP value $\bar{S}_t^{\delta^*} = \frac{S_t^{\delta^*}}{S_0^{\delta^*}}$ satisfies the SDE

$$d\bar{S}_t^{\delta^*} = \alpha_t^{\delta^*} dt + \sqrt{\alpha_t^{\delta^*} \bar{S}_t^{\delta^*}} d\tilde{W}_t, \quad (6.1)$$

where

$$\alpha_t^{\delta^*} = \bar{S}_t^{\delta^*} |\theta_t|^2 \quad (6.2)$$

is the *discounted GOP drift* and

$$d\tilde{W}_t = \sum_{k=1}^d \frac{\theta_t^k}{|\theta_t|} dW_t^k \quad (6.3)$$

is the differential of a standard Wiener process for $t \in [0, \infty)$. Note that $\alpha_t^{\delta^*}$ can form a very general stochastic process and the reparametrization of the dynamics

of the GOP given in the SDE (3.12) in the form of (6.1) creates no loss in generality. If the discounted GOP drift $\alpha_t^{\delta^*}$ is interpreted as the increase per unit of time of the *underlying value* of the discounted GOP, then this underlying value $\varphi(t)$ at time t is given as

$$\varphi(t) = \varphi(0) + \int_0^t \alpha_s^{\delta^*} ds \quad (6.4)$$

for $t \in [0, \infty)$. Now, if one interprets the underlying value as a transformed time, then \bar{S}^{δ^*} is a *squared Bessel process of dimension four*, when observed in this φ -time. This result is general, since we have not imposed any major modeling assumptions on the given continuous financial market. Therefore, the dynamics of the discounted market portfolio, when viewed in the transformed time scale that is naturally observable as we will see below, follows a very particular diffusion process. Fortunately, this process possesses a well-known transition density, see Revuz & Yor (1999), which is as explicit as that of the Black-Scholes model.

It is now of interest to study the transformed time. To observe the φ -time let us consider the square root of the discounted GOP, which satisfies the SDE

$$d\sqrt{\bar{S}_t^{\delta^*}} = \frac{3\alpha_t^{\delta^*}}{8\sqrt{\bar{S}_t^{\delta^*}}} dt + \frac{1}{2}\sqrt{\alpha_t^{\delta^*}} d\tilde{W}_t \quad (6.5)$$

for $t \in [0, \infty)$, as a consequence of (6.1) and the Itô formula. The quadratic variation of $\sqrt{\bar{S}^{\delta^*}}$ equals

$$\left\langle \sqrt{\bar{S}^{\delta^*}} \right\rangle_t = \frac{1}{4}(\varphi(t) - \varphi(0)) \quad (6.6)$$

for $t \in [0, \infty)$. This means that we can observe increments $\varphi(t) - \varphi(0)$ of the transformed time via the formula

$$\varphi(t) - \varphi(0) = 4 \left\langle \sqrt{\bar{S}^{\delta^*}} \right\rangle_t \quad (6.7)$$

for $t \in [0, \infty)$. The slightly random curve in Figure 4 is the increment of the empirical transformed time $\varphi(t) - \varphi(0)$ for the same data as in Figure 3.

We note, in particular, the small variation of this curve with an in average exponential slope, as would be suggested by an exponentially growing world economy. To reflect this economically based feature in modeling, we make the following assumption:

Assumption 6.2 *The discounted GOP drift is an exponential function.*

In line with this assumption we model the drift of the discounted GOP as follows:

$$\alpha_t^{\delta^*} = \alpha_0 \exp\{\eta t\}, \quad (6.8)$$

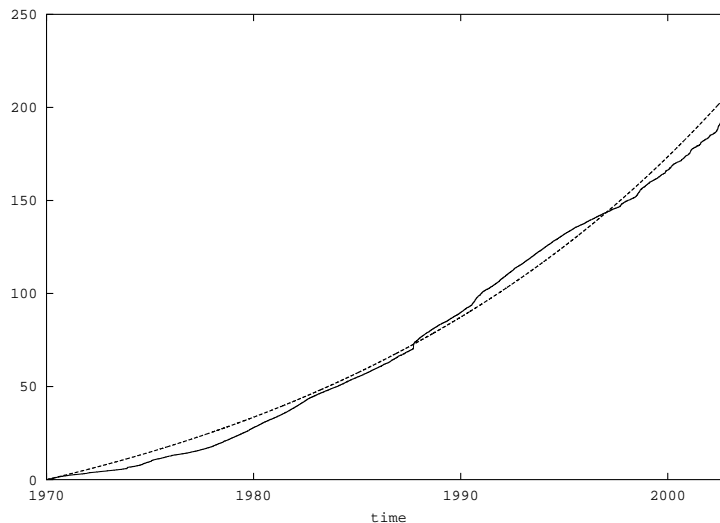


Figure 4: Empirical transformed time for discounted MSCI and theoretical transformed time.

where η can be interpreted as the *net growth rate* of the market portfolio. This yields a stylized version of the *minimal market model* (MMM), see Platen (2002, 2005b). More general versions of the MMM can be obtained by making the net growth rate time dependent or a stochastic process. Note that the smooth line in Figure 4 represents the theoretical transformed time with $\eta = 0.049$. This value for η has been estimated by Dimson, Marsh & Staunton (2002) for the world stock index when denominated in units of the US Dollar savings account, using data spanning the entire last century.

The volatility of the GOP obtains under the parametrization (6.8) of the discounted GOP drift the form

$$|\theta_t| = \sqrt{\frac{\alpha_t^{\delta_*}}{\bar{S}_t^{\delta_*}}} \quad (6.9)$$

for $t \in [0, \infty)$, by (3.12), (4.4) and (6.8). In Figure 5, using the MSCI as a proxy for the GOP, we plot the GOP volatility for the same empirical data as before, when setting $\alpha_0 = 10.5$ and $\eta = 0.049$. We remark that the resulting Student t distributed log-returns of the MMM with degrees of freedom four match extremely well the observed log-returns of market indices, see Markowitz & Usmen (1996) and Fergusson & Platen (2005).

According to (6.9) the GOP volatility equals the total market price of risk, which is one of the inputs for the Markowitz efficient frontier. By using the historical US short rate and the total market price of risk, that is the volatility of the MSCI, one obtains the evolution of the efficient frontier over time. In Figure 6 we show the efficient frontier for the US market from March to December 2000. The substantial movements of the efficient frontier make it understandable why authors such as Fama & French (2003) and many others have experienced extreme difficulties when trying to verify the CAPM empirically. The same applies to the

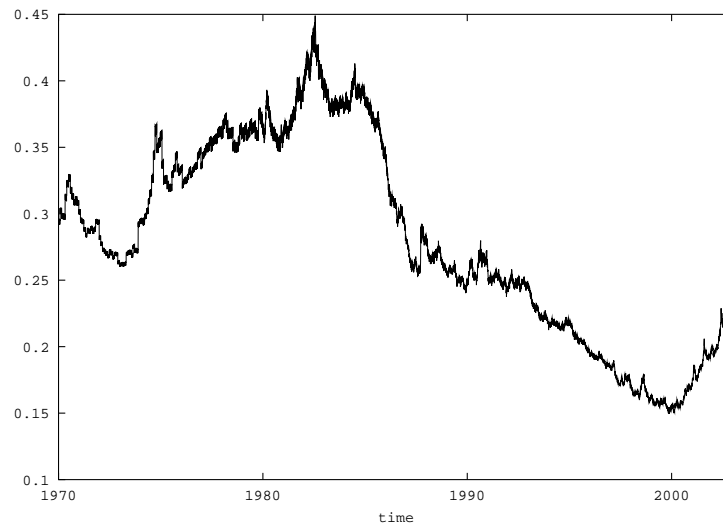


Figure 5: Volatility of the MSCI under the MMM.

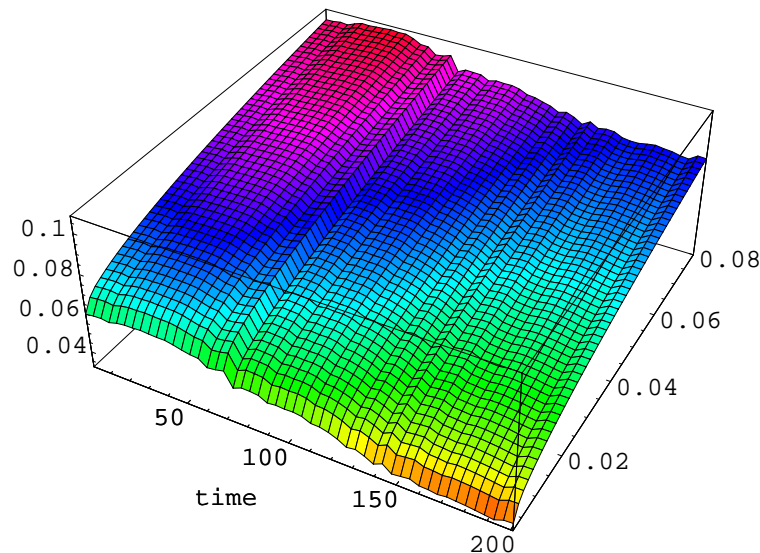


Figure 6: Efficient frontier over time.

extensive literature that tries to estimate the obviously strongly fluctuating betas of particular securities and portfolios. The methodology described above allows one to obtain at a given time reasonable, but over time substantially fluctuating beta estimates by using volatilities and covariations, which will be demonstrated in forthcoming work.

7 Utility Maximization

Given the optimal dynamics of the GOP, as obtained in the previous section, we can now reconcile the above presented approach with the widely established concept of expected utility maximization. We identify the class of portfolios that

expected utility maximizers will form. At first, let us call a portfolio S^δ *fair* if equality holds in relation (2.5), see Platen (2002). Furthermore, we call a portfolio that is expressed in units of the GOP a *benchmarked portfolio*. A benchmarked value of a fair portfolio is then the best forecast of its future benchmarked values. It is reasonable to assume that investors form fair portfolios with their total investable wealth since these are the minimal portfolios that replicate given future payoffs, see Platen (2002).

Now, we consider a general *utility function* $U : [0, \infty) \rightarrow \Re$ which possesses a derivative U' that can be inverted. Taking again the time value of money into account, we consider an investor that maximizes the expected utility from discounted terminal wealth, that is

$$E(U(\bar{S}_T^\delta)) \rightarrow \max, \quad (7.1)$$

where the maximum is taken over the set of strictly positive, discounted fair portfolios \bar{S}^δ . Under the previous assumptions it has been shown in Platen (2005b) that the resulting *expected utility maximizing portfolio is an optimal portfolio*. Since reasonable utility functions have a derivative that is invertible, this means that two fund separation applies also for an expected utility maximizing portfolio. When one uses the notion of a risk aversion coefficient J_t^δ , given in (4.13), for parametrization of the above expected utility maximizing portfolio S^δ , then one gets according to Platen (2005b) the risk aversion coefficient

$$J_t^\delta = \frac{1}{1 - \frac{\hat{S}_t^0}{\hat{u}(t, \hat{S}_t^0)} \frac{\partial \hat{u}(t, \hat{S}_t^0)}{\partial \hat{S}_t^0}} \quad (7.2)$$

at time $t \in [0, T]$, where

$$\hat{u}(t, \hat{S}_t^0) = E\left(U'^{-1}(\lambda \hat{S}_T^0) \hat{S}_T^0 \mid \mathcal{A}_t\right) \quad (7.3)$$

with some constant $\lambda \in (-\infty, \infty)$. Here U'^{-1} denotes the inverse of U' and $\hat{S}_t^0 = \frac{S_t^0}{S_t^{\delta^*}}$ is the benchmarked savings account. Note that this result applies for an extremely wide range of utility functions under the previous general assumptions. Additionally, some widely used optimization objectives, such as *lower partial moments*, *semi-variance* and *weighted semi-variance* are covered by the above general result and lead to optimal portfolios for investors who prefer these types of investment targets.

The risk aversion coefficient (7.2) generalizes in some sense the Arrow-Pratt notion of absolute risk aversion, see Pratt (1964) and Arrow (1965). Obviously, the risk aversion coefficient of the market portfolio, which is that of the GOP, takes under the above assumptions the value $J_t^{\delta^*} = 1$, by (7.2) and (7.3).

In summary, as we have discussed previously, an investor with an extremely long investment time horizon should simply invest in the GOP, that is the market

portfolio. Any other investor has to find at any time his or her risk aversion coefficient by whatever method, for instance, using utility maximization, lower partial moments or just randomly. The characterization of the risk aversion coefficient determines then the corresponding optimal portfolio. It appears to be rational to expect usually a risk aversion coefficient $J_t^\delta \geq 1$. Otherwise the investor faces a larger risk than for another available optimal portfolio to achieve the same growth rate, which we have shown to be the prevailing investment indicator in the long run. In Platen (2004a) it has been shown that also in the short run the GOP cannot be systematically outperformed by any other positive portfolio.

Conclusion

The paper has discussed the question how investors with short and long term investment horizons should invest optimally. Under extremely general modeling assumptions it emphasizes the fact that investors with a sufficiently long term investment horizon should choose a growth optimal portfolio as investment portfolio. Short term investors are shown to find two fund separation optimal, investing a fraction of wealth in the growth optimal portfolio and the remainder in the savings account. Under the assumption that all investors in a global market prefer more rather than less wealth, the market portfolio turns out to be growth optimal. Furthermore, its discounted value follows a time transformed squared Bessel process of dimension four. When the corresponding time transformation is modeled with an exponential increase, then the minimal market model arises. This model has a known transition density and matches observed data very well. Finally, it has been pointed out that expected utility maximizers invest a fraction of their wealth in the growth optimal portfolio and the remainder in the savings account.

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